Acta Sci. Math., 48 (1985), 469—476

Multivariable composition of Sobol'ev functions

F. SZIGETI

Dedicated to Prof. K. Tandori on his 60th birthday

1. Introduction

In this paper we shall prove some theorems on the composition of a multivariable outer function and inner functions of one variable belonging to certain Sobol'ev spaces. These theorems are based on a generalization (see [7] and Assertion 1) of the result of F. RIESZ [4] and the well-known Sobol'ev embedding theorems. Very interesting special cases are considered when $W_2^s(\Omega):=H^s(\Omega)$. Really, in this case the spaces $H^s(\Omega)$ can be characterized by Fourier coefficients, thus a relation can be proved between the convergency rate of the Fourier coefficients of the components and that of the composition.

2. Results

The following main results will be proved.

Theorem 1. *Let* $n\in\mathbf{N}$, $c, d\in\mathbf{R}$ $(c<d)$, $p, q, r, s\in\mathbf{R}$. *Suppose that* $p, q, r\in]1, \infty[$ *and that the equality*

$$\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)=1-\frac{1}{r} \tag{1}$$

and the inequality

$$s>1+\frac{n-1}{p} \tag{2}$$

are satisfied. Then, for each $f\in W_p^s(\mathbf{R}^n)$ *and monotonous functions* $g_1, g_2, \ldots, g_n\in\in W_q^1(c, d)$, *the composition* $f\circ g$ *belongs to* $W_r^1(c, d)$.

Theorem 1 and the Sobol'ev embedding theorems give us the following

Received June 28, 1984.

Theorem 2. *Let* $n \in \mathbf{N}$, $c, d \in \mathbf{R}$ $(c<d)$, $p, q, r, s_1, s_2, s_3 \in \mathbf{R}$. *Suppose that* $p, q, r \in]1, \infty[$, $p, q \geqq r$, *and that the inequality*

$$s_0 := \left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)+\frac{1}{r} < 1 \tag{3}$$

is satisfied. Let $s_3 \in]s_0, 1]$,

$$s_1 \in \left[1+\frac{n-1}{p},\ \frac{n}{p}+\left(s_3-\frac{1}{r}\right)\left(1-\frac{1}{q}\right)^{-1}\right[, \tag{4}$$

$$s_2 \in \left[1,\ \frac{1}{q}+\left(s_3-\frac{1}{r}\right)\left(1-\frac{1}{p}\right)^{-1}\right[\tag{5}$$

be numbers satisfying the inequality

$$\left(s_1-\frac{n}{p}\right)\left(s_2-\frac{1}{q}\right) > s_3-\frac{1}{r}. \tag{6}$$

Then, for each $f \in W_p^{s_1}(\mathbf{R}^n)$ *and monotonous functions* $g_1, g_2, \ldots, g_n \in W_q^{s_2}(c, d)$, *the composition* $f \circ (g_1, g_2, \ldots, g_n)$ *belongs to* $W_r^{s_3}(c, d)$.

Now, we mention the special case of Theorem 2, when $p, q, r := 2$. We can use the following characterization by the Fourier transform and series:

a) $f \in H^s(\mathbf{R}^n)$ if and only if $x \mapsto \hat{f}(x)(1+\|x\|^2)^{s/2} \in L_2(\mathbf{R}^n)$,

b) $g \in H^s(c, d)$ if and only if $\sum_{n=1}^{\infty} |\hat{g}(n)|^2 n^{2s} < \infty$

where $\{\hat{g}(n)\colon n \in \mathbf{N}\}$ are the Fourier coefficients of the function $g \in L_2(c, d)$ with respect to the system $\{\varphi_n\colon n \in \mathbf{N}\}$, of the eigenfunctions of the eigenvalue problem

$$(-1)^k x^{(2k)} + x = \lambda x,$$

$$x^{(k)}(c) = x^{(k+1)}(c) = \ldots = x^{(2k-1)}(c) = 0, \quad x^{(k)}(d) = x^{(k+1)}(d) = \ldots = x^{(2k-1)}(d) = 0$$

induced by the immersion $H^k(c, d) \subset L_2(c, d)$ and $s \leqq k$. If $k := 1$, then $\varphi_1(t) = (d-c)^{-1}$,

$$\varphi_n(t) = 2(d-c)^{-1} \cos\left[(n-1)\pi(t-c)(d-c)^{-1}\right] \quad (n \geqq 2).$$

Theorem 3. *Let* $n \in \mathbf{N}$, $c, d \in \mathbf{R}$, $(c<d)$. *Suppose that the numbers*

$$s_3 \in \left]\frac{3}{4}, 1\right[, \quad s_1 \in \left]\frac{n+1}{2}, 2s_3+\frac{n-2}{2}\right[, \quad s_2 \in \left[1, 2s_3-\frac{1}{2}\right[$$

satisfy the inequality

$$\left(s_1-\frac{n}{2}\right)\left(s_2-\frac{1}{2}\right) > s_3-\frac{1}{2}. \tag{7}$$

Then, if

$$x \mapsto \hat{f}(x)(1+\|x\|^2)^{s_1/2} \in L_2(\mathbf{R}^n), \tag{8}$$

$$\sum_{n=1}^{\infty} |\hat{g}_i(n)|^2 n^{2s_2} < \infty \quad (i = 1, 2, \ldots, n) \tag{9}$$

and g_i $(i=1, 2, \ldots, n)$ *are monotonous, then*

$$\sum_{n=1}^{\infty} |\widehat{f \circ (g_1, \ldots, g_n)}(n)|^2 n^{2s_3} < \infty \tag{10}$$

also holds.

It is clear that in Theorem 2 the space $H^s(\mathbf{R}^n)$ can be replaced by $H^s(\Omega)$ for a bounded region $\Omega \subset \mathbf{R}^n$, such that the closure of the range R_g of $g=(g_1, \ldots, g_n)$ belongs to Ω. In this case for each $f \in H^s(\Omega)$ there exists a function $\tilde{f} \in H^s(\mathbf{R}^n)$ such that $\tilde{f} \circ (g_1, \ldots, g_n) = f \circ (g_1, \ldots, g_n)$ holds. For this it is enough to resctrict f to a region $R_g \subset \Omega_1 \subset \Omega$ such that Ω_1 is a "good" region having the extension property (see [1], [2]). Thus the relation (8) can be replaced by

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^2 n^{2s_1} < \infty;$$

where $\{\hat{f}(n)\colon n \in \mathbf{N}\}$ are the Fourier coefficients with respect to an orthonormed system of eigenfunctions of an elliptic problem related to the embedding $H^k(\Omega) \subset L_2(\Omega)$ where $s \leqq k$ holds (see [3], [5]).

For the spaces $W_p^s(\mathbf{R}^n)$ and $W_p^2(c, d)$ we can prove Theorem 4 because the order of differentiability is high enough.

Theorem 4. *Let* $n \in \mathbf{N}$, $c, d \in \mathbf{R}$ $(c<d)$, $p \in \mathbf{R}$. *Suppose that* $1<p$, $s>2+(n-1)/p$. *If* $f \in W_p^s(\mathbf{R}^n)$ *and the monotonous functions* $g_1, g_2, \ldots, g_n \in W_p^2(c, d)$ *then the composition* $f \circ (g_1, g_2, \ldots, g_n)$ *belongs to the same space* $W_p^2(c, d)$.

In Theorems 1—4 the monotony of the inner functions play a very important role. Finally we show a theorem, in which the monotony of the inner functions can be omitted.

Theorem 5. *Let* $n \in \mathbf{N}$, $c, d \in \mathbf{R}$ $(c<d)$, $p, q \in]1, \infty[$, $s \in \mathbf{R}$. *Suppose that* $s>1+(n/p)$. *Then, if* $f \in W_p^s(\mathbf{R}^n)$, $g_i \in W_q^1(c, d)$ $(i=1, \ldots, n)$, *then* $f \circ (g_1, \ldots, g_n) \in W_q^1(c, d)$.

3. Outline on Sobol'ev spaces

In this section we survey some facts on the Sobol'ev spaces.

Let $k, n\in\mathbf{N}$, $p\in]1, \infty[$, $\Omega\subset\mathbf{R}^n$ be an open subset. The Sobol'ev space $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega):=\{f\colon D^\alpha f\in L_p(\Omega),\ |\alpha|\leqq k\}$$

equipped with an appropriate norm (see [1, Ch. III]).

Let $W_p^0(\Omega):=L_p(\Omega)$.

If $s\in\mathbf{R}_+$, then the Sobol'ev space $W_p^s(\Omega)$ is defined by

$$W_p^s(\Omega):=\Big\{f\in W_p^{[s]}(\Omega)\colon \int\limits_{\Omega\times\Omega}\frac{|D^\alpha f(x)-D^\alpha f(y)|^p}{|x-y|^{n+(s-[s])p}}\,dx\,dy<\infty,\quad |\alpha|=[s]\Big\}$$

equipped again with an appropriate norm (see [1, Ch. VII]).

Here $[s]$ denotes the entire part of the real number s. The Sobol'ev embedding theorems will be used (see [1], [2]) in the next:

a) if $s>n/p$, then $W_p^s(\Omega)\subset C_B(\Omega)$ and the embedding is continuous and linear,

b) if $s_1, s_2\in\mathbf{R}$, $p_1, p_2\in]1, \infty[$, $s_1\leqq s_2$, $p_1\geqq p_2$ and $(n/p_1)-s_1\geqq (n/p_2)-s_2$, then $W_{p_1}^{s_1}(\Omega)\supset W_{p_2}^{s_2}(\Omega)$ and the embedding is continuous and linear.

The mentioned result of F. Riesz [4] is the following: an absolutely continuous function $f\colon]a, b[\to\mathbf{R}$ (or $\mathbf{C}$) has its derivative $f'\in L_p[a, b]$ if and only if there exists a number $K\geqq 0$ such that, for any system $\{]a_i, b_i[\subset]a, b[\colon i\in I\}$ of nonoverlapping bounded subintervals, the inequality

$$\sum_i\frac{|f(b_i)-f(a_i)|^p}{|b_i-a_i|^{p-1}}\leqq K$$

holds, and the best constant is $\|f'\|_{L_p}^p$.

As it was mentioned in the introduction, the proofs are based on the Riesz theorem and

Assertion 1. (see [7]). *Let $n\in\mathbf{N}$, $p, s\in\mathbf{R}$. Suppose that $p\in]1, \infty[$ and $s>1+((n-1)/p)$. If $f\in W_p^s(\mathbf{R}^n)$, then there exist real numbers $K_i\geqq 0$ $(i=1, 2, \dots, n)$ such that, for any integers $I_i\in\mathbf{N}$, systems*

$$\{]a_{ij}, b_{ij}[\subset\mathbf{R}\colon j=1, 2, \dots, I_i\}$$

of nonoverlapping bounded subintervals and sets $\{\xi_{ij}\in\mathbf{R}^{n-1}\colon j=1, 2, \dots, I_i\}$, the inequalities

$$\sum_{j=1}^{I_i}\frac{|f_{i,b_{ij}}(\xi_{ij})-f_{i,a_{ij}}(\xi_{ij})|^p}{|b_{ij}-a_{ij}|^{p-1}}\leqq K_i$$

hold, where for any $i=1, 2, \dots, n$, $a\in\mathbf{R}$ the function $f_{i,a}\colon \mathbf{R}^{n-1}\to\mathbf{R}$ (or $\mathbf{C}$) is defined by

$$\xi\mapsto f(\xi_1, \dots, \xi_{i-1}, a, \xi_i, \dots, \xi_{n-1}).$$

We mention that the best constants K_i $(i=1, 2, \ldots, n)$ are

$$\int_{\mathbf{R}} \|(\partial_i f)_{i,t}\|^p_{W_p^{s-1}(\mathbf{R}^{n-1})}\, dt.$$

Next, we need the following

Assertion 2 (see [6]). *Let* $c, d \in \mathbf{R}$ $(c<d)$ $p\in]1, \infty[$. *If* $f, g \in W_p^1(a, b)$, *then* $fg \in W_p^1(a, b)$.

The proof can be made by the Riesz theorem.

4. Proofs and remarks

Proof of Theorem 1. Define the number $\alpha := r(1-(1/p))$. We can easily see that (1) is equivalent to the equality

$$\frac{r}{p}+\frac{r}{q}\left(1-\frac{1}{p}\right) = 1. \tag{11}$$

Our proof if based on the Riesz theorem and Assertion 1. For this, let $I\in \mathbf{N}$, $\{]c_i, d_i[\subset]c, d[:\ i=1, 2, \ldots, I\}$ be a system of nonoverlapping subintervals. Let

$$\xi_{ij} := \big(g_1(d_i), \ldots, g_{j-1}(d_i),\ g_{j+1}(c_i), \ldots, g_n(c_i)\big) \quad (\in \mathbf{R}^{n-1}),$$
$$d_{ij} := g_j(d_i), \quad c_{ij} := g_j(c_i)$$

$(j=1, 2, \ldots, n, i=1, 2, \ldots, I)$. Then by (11) we can estimate with the Hölder inequality:

$$\sum_{i=1}^{I} \frac{|f\circ(g_1, \ldots, g_n)(d_i)-f\circ(g_1, \ldots, g_n)(c_i)|^r}{|d_i-c_i|^{r-1}} \leqq$$
$$\leqq \sum_{i=1}^{I} n^{r-1} \sum_{j=1}^{n} \frac{|f_{j,d_{ij}}(\xi_{ij})-f_{j,c_{ij}}(\xi_{ij})|^r}{|d_{ij}-c_{ij}|^{\alpha}} \cdot \frac{|g_j(d_i)-g_j(c_i)|^{\alpha}}{|d_i-c_i|^{r-1}} \leqq$$
$$\leqq n^{r-1} \sum_{j=1}^{n} \left(\sum_{i=1}^{I} \frac{|f_{j,d_{ij}}(\xi_{ij})-f_{j,c_{ij}}(\xi_{ij})|^p}{|d_{ij}-c_{ij}|^{\alpha(p/r)}} \right)^{r/p} \left(\sum_{i=1}^{I} \frac{|g_j(d_i)-g_j(c_i)|^q}{|d_i-c_i|^{(r-1)(q/\alpha)}} \right)^{\alpha/q}.$$

We can check that $\alpha(p/r)=p-1$, $(r-1)(q/\alpha)=q-1$, $\alpha/q=(r/q)(1-(1/p))$ thus, by Assertion 1 and the Riesz theorem

$$\sum_{i=1}^{I} \frac{|f\circ(g_1, \ldots, g_n)(d_i)-f\circ(g_1, \ldots, g_n)(c_i)|^r}{|d_i-c_i|^{r-1}} \leqq n^{r-1}\Big(\sum_{j=1}^{n} \|\partial_j f\|^{p(r/p)}_{W_p^{s-1}} \|g_j'\|^{r(1-(1/p))}_{L_q}\Big),$$

because

$$\int_{\mathbf{R}} \|(\partial_j f)_{j,t}\|^p_{W_p^{s-1}(\mathbf{R}^{n-1})}\, dt \leqq \|\partial_j f\|^p_{W_p^{s-1}(\mathbf{R}^n)};$$

that is

$$\|(f\circ(g_1,\dots,g_n))'\|_{L_r} \leqq n^{(r-1)/r}\Big(\sum_{j=1}^{n}\|\partial_j f\|^r_{W_p^{s-1}}\|g_j'\|^{r(1-(1/p))}_{L_q}\Big)^{1/r} \leqq$$

$$\leqq n^{(r-1)/r}\|f\|_{W_p^s}\Big(\sum_{j=1}^{n}\|g_j'\|^{r(1-(1/p))}_{L_q}\Big)^{1/r}.$$

Proof of Theorem 2. From (3) the inequalities

$$\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right) < s_3-\frac{1}{r} \leqq 1-\frac{1}{r}$$

follow immediately. Thus the inequalities

$$1-\frac{1}{p} \leqq s_1-\frac{n}{p} < \left(s_3-\frac{1}{r}\right)\left(1-\frac{1}{q}\right)^{-1} \leqq \left(1-\frac{1}{r}\right)\left(1-\frac{1}{q}\right)^{-1} \leqq 1,$$

$$1-\frac{1}{q} \leqq s_2-\frac{1}{q} < \left(s_3-\frac{1}{r}\right)\left(1-\frac{1}{p}\right)^{-1} \leqq \left(1-\frac{1}{r}\right)\left(1-\frac{1}{p}\right)^{-1} \leqq 1$$

hold, so the intervals (4), (5) are nonempty, and the inequality $1\leqq s_2<1+(1/q)$ is satisfied. Thus there exist numbers $p_0\geqq p$, $q_0\geqq q$, $r_0\leqq r$ such that for p_0, q_0, r_0

$$\text{(12)}\qquad s_1-\frac{n}{p}=:1-\frac{1}{p_0},\quad s_2-\frac{1}{q}=:1-\frac{1}{q_0},\quad \left(s_1-\frac{n}{p}\right)\left(s_2-\frac{1}{q}\right)=:1-\frac{1}{r_0}$$

hold. By theorem 1, if $f\in W^{s_1}_{p_0}(\mathbf{R}^n)$ and the monotonous functions $g_i\in W^1_{q_0}(c,d)$ $(i=1,\dots,n)$, then the composition $f\circ(g_1,\dots,g_n)$ belongs to $W^1_{r_0}(c,d)$. By (12) the $W^{s_2}_q(c,d)\subset W^1_{q_0}(c,d)$ and $W^{s_3}_r(c,d)\supset W^1_{r_0}(c,d)$ embeddings are continuous and linear, so, if $f\in W^{s_1}_p(\mathbf{R}^n)$ and the monotonous functions $g_i\in W^{s_2}_q(c,d)$ $(i=1,\dots,n)$, then $f\in W^{s_1}_{p_0}(\mathbf{R}^n)$ $(p=p_0)$, $g_i\in W^1_{q_0}(c,d)$, thus $f\circ(g_1,\dots,g_n)\in W^1_{r_0}(c,d)\subset W^{s_3}_r(c,d)$ too.

Theorem 3 follows from Theorem 2 and the characterizations a), b) (see Section 2).

Proof of Theorem 4. By assumption $f\in W^s_p(\mathbf{R}^n)$, and $g_i\in W^2_p(c,d)$ $(i=1,2,\dots,n)$, so by the chain rule

$$\text{(14)}\qquad (f\circ(g_1,g_2,\dots,g_n))' = \sum_{i=1}^{n}\partial_i f\circ(g_1,g_2,\dots,g_n)g_i'.$$

Again by assumption the functions $g_i\in W^2_p(c,d)$ are monotonous and clearly $\partial_i f\in W^{s-1}_p(\mathbf{R}^n)$ $s-1>1+((n-1)/p)$. Thus by Theorem 2 $\partial_i f\circ(g_1,g_2,\dots,g_n)\in$ $\in W^1_p(c,d)$. On the other hand $g_i'\in W^1_p(c,d)$ holds obviously, so by Assertion 2 the sum (14) of products belongs to $W_p^1(c, d)$. This means that the composition

$$f\circ(g_1, g_2, \dots, g_n)\in W_p^2(c, d).$$

Remark 1. In the special case $p:=2$ we can use the characterization of the spaces of type H^s in terms of the Fourier coefficients. If $\Omega\subset\mathbf{R}^n$ and $(c, d)\subset\mathbf{R}$ are bounded, $f\in L_2(\Omega)$, the monotonous functions $g_i\in L_2(c, d)$ $(i=1, 2, \dots, n)$ and there exists the composition $f\circ(g_1, \dots, g_n)$, then from

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^2 n^{2s} < \infty,$$

$$\sum_{n=1}^{\infty} |\hat{g}_i(n)|^2 n^4 < \infty \quad (i = 1, \dots, n)$$

follows that

$$\sum_{n=1}^{\infty} |\widehat{f\circ(g_1, \dots, g_n)}(n)|^2 n^4 < \infty.$$

The Fourier coefficients relate to the above mentioned orthonormed systems of eigenfunctions.

Proof of Theorem 5. Apply the Sobol'eb embedding theorem for $W_p^s(\mathbf{R}^n)\subset \subset C^1(\mathbf{R}^n)$. Thus by the chain rule (14) is satisfied almost everywhere over (c, d). Let M denote the norm of the embedding $W_p^{s-1}(\mathbf{R}^n)\subset C_B(\mathbf{R}^n)$. Then

$$\|(f\circ(g_1, \dots, g_n))'\|_{L_q} = \|\sum_{i=1}^{n} \partial_i f\circ(g_1, \dots, g_n)g_i'\|_{L_q} \leqq$$

$$\leqq \sum_{i=1}^{n} \Big(\int_c^d |\partial_i f\circ(g_1, \dots, g_n)|^q |g_i'|^q\Big)^{1/q} \leqq$$

$$\leqq \sum_{i=1}^{n} \|\partial_i f\|_{C_B(\mathbf{R}^n)} \|g_i'\|_{L_q} \leqq M \sum_{i=1}^{n} \|\partial_i f\|_{W_p^{s-1}} \|g_i'\|_{L_q} \leqq$$

$$\leqq M\|f\|_{W_p^s} \sum_{i=1}^{n} \|g_i'\|_{L_q}.$$

Thus $f\circ(g_1, \dots, g_n)\in W_q^1(c, d)$.

Remark 2. The inequality

$$\|(f\circ(g_1, \dots, g_n))'\|_{L_q} \leqq M\|f\|_{W_p^s} \sum_{i=1}^{n} \|g_i'\|_L \tag{15}$$

also is obtained.

References

[1] R. A. Adams, *Sobol'ev Spaces,* Academic Press (New York—San Francisco—London, 1975).
[2] O. V. Besov, V. P. Il'in and S. M. Nikol'skij, *Integral representations of functions and embedding theorems,* Nauka (Moskva, 1975). (in Russian)
[3] J. L. Lions et E. Magenes, *Problemes aux limites non homogenes et applications. 1,* Dunod (Paris, 1968).
[4] F. Riesz, Untersuchungen über Systeme integrierbarer Functionen, *Math. Ann.,* 69 (1910), 449—497.
[5] F. Riesz et B. Sz. Nagy, *Lecons d'Analyse fonctionelle,* Akadémiai Kiadó (Budapest, 1952).
[6] F. Szigeti, Composition of Sobol'ev functions and applications. I, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.,* to appear.
[7] F. Szigeti, Necessary conditions for certain Sobol'ev spaces, *Acta Math. Hung.,* to appear.

LORÁND EÖTVÖS UNIVERSITY
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY